

# BANACH SPACES EMBEDDING INTO $L_0$

BY

N. J. KALTON<sup>†</sup>*Department of Mathematics, University of Missouri — Columbia, Columbia, MO 65211, USA*

## ABSTRACT

Our main result in this paper is that a Banach space  $X$  embeds into  $L_1$  if and only if  $l_1(X)$  embeds into  $L_0$ ; more generally if  $1 \leq p < 2$ ,  $X$  embeds into  $L_p$  if and only if  $l_p(X)$  embeds into  $L_0$ .

## 1. Introduction

It is still unknown whether every Banach space which embeds into  $L_0 = L_0(0, 1)$  is isomorphic to a subspace of  $L_1$ . This problem was suggested by Kwapien [9]. Our main result in this paper is that a Banach space  $X$  embeds into  $L_1$  if and only if  $l_1(X)$  embeds into  $L_0$ ; more generally if  $1 \leq p < 2$ ,  $X$  embeds into  $L_p$  if and only if  $l_p(X)$  embeds into  $L_0$ .

Before discussing the problem and the contents of the paper, we introduce some notation. Throughout the paper  $\Omega$  will denote a compact metric space,  $\Sigma$  the  $\sigma$ -algebra of Borel subsets of  $\Omega$  and  $P$  a nonatomic probability measure on  $\Sigma$ . Of course there is no loss of generality in taking  $\Omega = [0, 1]$  and  $P$  Lebesgue measure on  $[0, 1]$ . For  $0 \leq p < \infty$ ,  $L_p(\Omega, \Sigma, P)$  will be abbreviated to  $L_p$ . We also denote by  $L(p, \infty)$  the Lorentz space, weak  $L_p$ , of all  $f \in L_0(\Omega, \Sigma, P)$  so that

$$\|f\|_{p, \infty} = \sup_{0 < x < \infty} x(P(|f| > x))^{1/p} < \infty.$$

Let us say that a linear operator  $V: X \rightarrow L_p$  (or  $V: X \rightarrow L(p, \infty)$ ) is a *strong embedding* if  $V$  is an isomorphism onto its range, and the topology of  $L_p$  (or  $L(p, \infty)$ ) on its range coincides with the  $L_0$ -topology (convergence in measure).

For  $1 \leq p \leq 2$ , a Banach space  $X$  is said to be of type  $p$  (Rademacher) if there

<sup>†</sup> Research supported by NSF grant MCS-8301099.

Received November 1, 1984 and in revised form March 31, 1985

is a constant  $C$  so that

$$\text{Avg} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^p \leq C \sum_{i=1}^n \|x_i\|^p$$

where the average is taken over all choices of sign  $\varepsilon_i = \pm 1$ . Every Banach space is of type one.

The first progress on Kwapien’s problem was made by Nikishin [14] who showed that a Banach subspace of  $L_0$  is in fact isomorphic and is a subspace of  $L_p$  for every  $p < 1$ . Later [15] (cf. expositions in [13] and [5]) he refined this result to establish the following factorization theorem:

**THEOREM 1.1. (Nikishin).** *Let  $X$  be a Banach space of type  $p$  ( $1 \leq p < 2$ ) and let  $V : X \rightarrow L_0$  be any continuous linear operator. Then given  $\varepsilon > 0$ , there exists a set  $E$  with  $P(E) \geq 1 - \varepsilon$  so that if*

$$Wx = 1_E Vx$$

*then  $W$  is a bounded linear operator from  $X$  into  $L(p, \infty)$ .*

**COROLLARY 1.2.** *Every Banach subspace of  $L_0$  can be strongly embedded in  $L(1, \infty)$ .*

**COROLLARY 1.3.** *Every Banach subspace of  $L_0$  of type  $p$  can be strongly embedded in  $L(p, \infty)$ .*

We obtain the results announced in the introduction by a close analysis of the spaces  $L(p, \infty)$ . Our methods hinge on the existence of non-trivial continuous linear functionals on the non-locally convex quasi-Banach space  $L(1, \infty)$ . This fact was first observed by Cwikel and Sagher [2] and recently the dual of  $L(1, \infty)$  has been studied by Cwikel and Fefferman [1] and by Kupka and Peck [8]. Our methods are quite similar to techniques in [8] but were obtained independently.

In fact for convenience, in Section 2, we study not  $L(p, \infty)$  but instead the  $l_\infty$ -product  $l_\infty(L(p, \infty))$  which we abbreviate to  $\mathcal{Y}_p$ . Thus  $\mathcal{Y}_p$  consists of all sequences  $F = (f_n)$  where  $f_n \in L(p, \infty)$  and

$$\|F\| = \sup_n \|f_n\|_{p, \infty} < \infty.$$

We do, however, give an application of these ideas to  $L(p, \infty)$  showing the standard embedding of  $L_p$  into  $L(p, \infty)$  using  $p$ -stable processes yields a complemented subspace.

In Section 3 we give our applications to Banach subspaces of  $L_0$  and in Section 4 we make our closing remarks.

We thank B. Maurey and G. Pisier for helpful comments.

**2. An explicit linear functional on  $\mathcal{U}_1$**

Let  $\mathcal{U}$  be any ultrafilter on  $(2, \infty)$  which includes each of the sets  $(x, \infty)$  for  $x > 2$ . Let  $\mathcal{V}$  be any ultrafilter on  $\mathbb{N}$  containing each of the sets  $\{m \in \mathbb{N} : m \geq n\}$  for  $n \in \mathbb{N}$ .

For  $f \in L_0$  and  $x \geq 0$  we define the truncation  $\tau_x f$  by

$$\begin{aligned} \tau_x f(\omega) &= f(\omega) && \text{if } |f(\omega)| \leq x, \\ &= x && \text{if } f(\omega) > x, \\ &= -x && \text{if } f(\omega) < -x. \end{aligned}$$

Now suppose  $F = (f_n) \in \mathcal{U}_1$ . We define

$$\phi(F) = \lim_{x \in \mathcal{U}} \frac{1}{\log x} \lim_{n \in \mathcal{V}} \mathcal{E}(\tau_x f_n),$$

where  $\mathcal{E}$  denotes expectation.

LEMMA 2.1.  $\phi$  is a positive linear functional on  $\mathcal{U}_1$  and  $\|\phi\| = 1$ .

PROOF. First we note that  $\phi$  is well-defined and  $\|\phi(F)\| \leq \|F\|$  for  $F \in \mathcal{U}_1$ . In fact if  $f \in L(1, \infty)$ , with  $\|f\| \leq 1$ ,

$$|\mathcal{E}(\tau_x f)| \leq 1 + \log x, \quad x \geq 1$$

and hence  $\phi$  is well-defined and

$$|\phi(F)| \leq \|F\|.$$

Now we observe that

- (1)  $\phi(F + G) = \phi(F) + \phi(G), \quad |F| \wedge |G| = 0;$
- (2)  $\phi(F) \leq \phi(G), \quad \text{whenever } F \leq G;$
- (3)  $\phi(-F) = -\phi(F), \quad F \in \mathcal{U}_1.$

Now suppose  $0 < \alpha < 1$ . If  $f \in L(1, \infty)$ ,

$$\begin{aligned} \mathcal{E}(|\tau_x(\alpha f) - \tau_{\alpha x}(\alpha f)|) &\leq (x - \alpha x)P(|f| > x) \\ &\leq (1 - \alpha)\|f\|. \end{aligned}$$

Hence it follows easily that

$$\begin{aligned} \phi(\alpha F) &= \lim_{x \in \mathbb{Q}_+} \frac{1}{\log x} \lim_{n \in \mathbb{Y}} \mathcal{E}(\tau_{\alpha x}(\alpha f_n)) \\ &= \alpha \phi(F). \end{aligned}$$

We conclude that, using (3),

$$(4) \quad \phi(\alpha F) = \alpha \phi(F), \quad \alpha \in \mathbf{R}, \quad F \in \mathcal{Y}_1.$$

Now suppose  $F, G \geq 0$  and let  $H = F + G$ . Fix  $N \in \mathbf{N}$ . Then for  $k = 1, 2, \dots, N - 1$  and  $n \in \mathbf{N}$  we set

$$A_{kn} = \left\{ \omega \in \Omega : \frac{k-1}{N} h_n(\omega) \leq f_n(\omega) < \frac{k}{N} h_n(\omega) \right\}.$$

For  $k = N$  we set

$$A_{Nn} = \left\{ \omega \in \Omega : \frac{N-1}{N} h_n(\omega) \leq f_n(\omega) \leq h_n(\omega) \right\}.$$

For each  $n \in \mathbf{N}$ ,  $A_{1n}, \dots, A_{Nn}$  partitions  $\Omega$  into  $\Sigma$ -measurable sets and

$$\begin{aligned} \frac{k-1}{N} h_n 1_{A_{kn}} &\leq f_n 1_{A_{kn}} \leq \frac{k}{N} h_n 1_{A_{kn}}, \\ \frac{N-k}{N} h_n 1_{A_{kn}} &\leq g_n 1_{A_{kn}} \leq \frac{N-k+1}{N} h_n 1_{A_{kn}}. \end{aligned}$$

Now using properties (1), (2) and (4)

$$\frac{N-1}{N} \phi(H) \leq \phi(F) + \phi(G) \leq \frac{N+1}{N} \phi(H)$$

so  $\phi(F) + \phi(G) = \phi(H)$ . It follows easily that  $\phi$  is linear.

By considering any  $f$  so that  $P(f \geq x) = x^{-1}$  for  $x \geq 1$  we see that  $\|\phi\| = 1$ .

Note that there is an embedding  $J : L(1, \infty) \rightarrow \mathcal{Y}_1$  given by  $J(f) = (f, f, \dots)$ . Let

$$\phi_0(f) = \phi \circ J(f) = \lim_{x \in \mathbb{Q}_+} \frac{1}{\log x} \mathcal{E}(\tau_x f)$$

which is a positive linear functional on  $L(1, \infty)$  with  $\|\phi_0\| = 1$  (compare [8]).

LEMMA 2.2. *Suppose  $1 < p < \infty$  and  $1/p + 1/q = 1$ . If  $F \in \mathcal{Y}_p$  and  $G \in \mathcal{Y}_q$  then  $FG \in \mathcal{Y}_1$  and*

$$|\phi(FG)| \leq (\phi(|F|^p))^{1/p} (\phi(|G|^q))^{1/q}.$$

PROOF. Simply note that

$$|FG| \leq \frac{|F|^p}{p} + \frac{|G|^q}{q}$$

so that  $FG \in \mathcal{Y}_1$  and

$$\phi(|FG|) \leq \frac{1}{p} \phi(|F|^p) + \frac{1}{q} \phi(|G|^q).$$

In particular if  $\phi(|F|^p) \leq 1$ ,  $\phi(|G|^q) \leq 1$  then  $\phi(|FG|) \leq 1$  and the lemma follows by homogeneity.

LEMMA 2.3. *If  $1 \leq p < \infty$  then  $F \rightarrow (\phi(|F|_p))^{1/p}$  is a semi-norm on  $\mathcal{Y}_p$ .*

PROOF. For  $p = 1$  this is clear. For  $p > 1$  note that

$$\begin{aligned} \phi(|F + G|^p) &\leq \phi(|F| \cdot |F + G|^{p-1}) + \phi(|G| \cdot |F + G|^{p-1}) \\ &\leq ((\phi(|F|^p))^{1/p} + \phi(|G|^p)^{1/p}) \phi(|F + G|^{p-1}), \end{aligned}$$

where  $1/p + 1/q = 1$ . The lemma follows.

On  $\mathcal{Y}_p$  let

$$\| \| F \| \| = \phi(|F|^p)^{1/p}.$$

Then  $\| \| \cdot \| \|$  is a lattice semi-norm on  $\mathcal{Y}_p$  satisfying

$$\| \| F + G \| \| ^p = \| \| F \| \| ^p + \| \| G \| \| ^p, \quad |F| \wedge |G| = 0.$$

Let  $M_p = \{F \in \mathcal{Y}_p : \| \| F \| \| _p = 0\}$ . Then with the quotient norm  $\mathcal{Y}_p/M_p$  is a normed lattice whose completion is an  $AL_p$ -space (see Lacey [11] Chapter 5). We denote this space by  $\mathcal{X}_p$  and denote by  $K_p$  the induced map  $K_p : \mathcal{Y}_p \rightarrow \mathcal{X}_p$ .

We conclude this section with an elementary application of these ideas. We denote by  $J_p$  the natural map  $J_p : L(p, \infty) \rightarrow \mathcal{Y}_p$  given by  $J_p(f) = (f, f, \dots)$ . Note that

$$\| \| K_p J_p f \| \| = (\phi_0(|f|^p))^{1/p}$$

where  $\phi_0 = \phi J_1$  as above.

We recall (cf. [7]) that for  $1 \leq p \leq 2$  there is an embedding of  $L_p$  into  $L(p, \infty)$  using a  $p$ -stable process. Precisely, there is a linear map  $V_p : L_p \rightarrow L(p, \infty)$  so that

- (i)  $V_p f$  and  $V_p g$  are independent whenever  $|f| \wedge |g| = 0$ .
- (ii)  $\mathcal{E}(\exp(itV_p f)) = \exp(-|t|^p \| \| f \| \| _p^p)$ .

THEOREM 2.5. *The range of  $V_p$  is complemented in  $L(p, \infty)$  for  $1 \leq p < 2$ .*

PROOF. Note that for  $f \in L(p, \infty)$

$$\begin{aligned}\phi_0(|f|^p) &= \lim_{x \in \mathbb{Q}_+} \frac{1}{\log x} \mathcal{E}(\tau_x |f|^p) \\ &= \lim_{x \in \mathbb{Q}_+} \frac{1}{\log x} \int_0^x P(|f|^p > t) dt \\ &= \lim_{x \rightarrow \infty} xP(|f|^p > x)\end{aligned}$$

whenever this limit exists.

Now if  $\|f\|_p = 1$ ,  $\lim_{x \rightarrow \infty} xP(|V_p f|^p > x) = C_p$  (see [17]) where

$$C_p = \left[ \int_0^\infty \frac{\sin x}{x^p} dx \right]^{-1}.$$

Thus for  $f \in L_p$ ,

$$\phi_0(|V_p f|^p)^{1/p} = C_p \|f\|_p.$$

Hence  $K_p J_p V_p(L_p)$  is an isometric copy of  $L_p$  contained in  $\mathcal{X}_p$ . This implies that there is a projection of norm one of  $\mathcal{X}_p$  onto  $K_p J_p V_p(L_p)$  (this result goes back to Pelczynski [16] for the case  $p > 1$ ; see also Dor [3]).

Now let  $W : K_p J_p V_p(L_p) \rightarrow V_p(L_p)$  be the inverse of  $K_p J_p$ . Then  $W K_p J_p$  is the required projection.

REMARK. In the case  $p = 1$ ,  $\|W\| = 1$  and so the projection has norm one. In fact if  $\|f\| = 1$  then

$$P(|V_1 f| > x) = \frac{2}{\pi} \tan^{-1} \frac{1}{x}$$

so that

$$\begin{aligned}\|V_1 f\| &= \sup_{x > 0} \frac{2}{\pi} x \tan^{-1} \frac{1}{x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{\pi} x \tan^{-1} \frac{1}{x} \\ &= \frac{2}{\pi}\end{aligned}$$

and

$$\phi_0(|V_1 f|) = \|K_1 J_1 V_1 f\| = 2\pi^{-1}$$

so that  $K_1 J_1$  is an isometry on  $V_1(L_1)$ .

### 3. Applications

The following proposition will be useful later on. Although we know of no explicit reference for it, we believe that it is already known and that it is due to Maurey.

PROPOSITION 3.1. *Suppose  $1 < p < 2$ , and  $X$  is a Banach space so that  $l_p(X)$  embeds into  $L_0$ . then  $X$  (and  $l_p(X)$ ) are of type  $p$ .*

PROOF. We shall regard  $l_p(X)$  as a tensor product of  $l_p$  and  $X$ , and denote by  $(e_n)$  the standard vector basis of  $l_p$ .

We may assume that we have a strong embedding  $V: l_p(X) \rightarrow L_q$  where  $0 < q \leq 1$ . It follows that there are constants  $M < \infty$  and  $\varepsilon > 0$  so that

$$(1) \quad \|Vu\|_q \leq M\|u\|, \quad u \in l_p(X),$$

$$(2) \quad \|Vu \cdot 1_F\|_q \geq M^{-1}\|u\|, \quad u \in l_p(X),$$

whenever  $P(F) \geq 1 - \varepsilon$ .

We shall need two further facts. Suppose  $\{\varepsilon_n\}$  is a sequence of independent Bernoulli random variables on some probability space  $(\Omega', \Sigma', P')$  where  $P'\{\varepsilon_n = 1\} = P'\{\varepsilon_n = -1\} = \frac{1}{2}$ . Then there is a constant  $\gamma > 0$  so that for  $f_1, \dots, f_n \in L_q$

$$\gamma \|(\sum |f_i|^2)^{1/2}\|_q \leq (\mathcal{E}'(\|\sum \varepsilon_i f_i\|^q))^{1/q} \leq \|(\sum |f_i|^2)^{1/2}\|_q.$$

This is a simple deduction from Khintchine's inequality (see [12] p. 50).

Secondly we observe that  $L(p/2, \infty)$  is a  $p/2$ -normable; see [5] for a proof, although the result was known earlier [13]. Hence there is a constant  $C_0$  so that for  $f_1, \dots, f_n \in L(p, \infty)$

$$\|(\sum |f_i|^2)^{1/2}\|_{p,\infty} \leq C_0 \left( \sum_{i=1}^n \|f_i\|_{p,\infty}^p \right)^{1/p}.$$

To prove this simply note  $\|\sum |f_i|^2\|_{p/2,\infty} \leq C_1 (\sum \| |f_i|^2 \|_{p/2,\infty}^{p/2})^{2/p}$ . Hence as the injection  $L(p, \infty) \rightarrow L_q$  is continuous we have a constant  $C_1$  so that for  $f_1, \dots, f_n \in L(p, \infty)$ ,

$$\|(\sum |f_i|^2)^{1/2}\|_q \leq C_1 \left( \sum_{i=1}^n \|f_i\|_{p,\infty}^p \right)^{1/p}.$$

Combining these remarks we see that there is a constant  $C$  so that for  $f_1, \dots, f_n \in L(p, \infty)$

$$(\mathcal{E}'(\|\sum \varepsilon_i f_i\|_q^q))^{1/q} \leq C \left( \sum_{i=1}^n \|f_i\|_{p,\infty}^p \right)^{1/p}.$$

Fix  $A = (2/\varepsilon)^{1/q} M/\gamma$ .

Now let us suppose  $w_1, \dots, w_n \in X$  with  $\|w_i\| = 1$ . Set

$$f_{ij} = V(e_j \otimes w_i)$$

for  $j = 1, 2, \dots$  and  $1 \leq i \leq n$ .

We now select a sequence of disjoint sets  $E_j \in \Sigma$  by induction.

For convenience let  $E_0 = \emptyset$ . If  $E_k$  have been selected for  $k < j$ , we let  $F = E_0 \cup \dots \cup E_{j-1}$ . For each  $\delta > 0$  and  $1 \leq i \leq n$ , let

$$G_i(\delta) = \{\omega : |f_{ij}(\omega)| \geq A\delta^{-1/p} \text{ and } \omega \notin F\}.$$

Let  $G_i(0) = \emptyset$  for  $1 \leq i \leq n$ .

Let

$$\rho(\delta) = \max_{1 \leq i \leq n} P(G_i(\delta)).$$

Pick  $\delta_j = \sup\{\delta : \rho(\delta) \geq \delta\}$ . Since  $\rho$  is increasing  $\rho(\delta_j) \geq \delta_j$ , and so there is a set  $E_j$  disjoint from  $F$  and  $k(j)$  with  $1 \leq k(j) \leq n$  so that

(1)  $|f_{k(j),j}(\omega)| \geq A\delta_j^{-1/p}$ ,  $\omega \in E_j$ .

(2) If  $H \in \Sigma$  is disjoint from  $E_1, \dots, E_{j-1}$  and if  $1 \leq i \leq n$  is such that

$$|f_{ij}(\omega)| \geq AP(H)^{-1/p}, \quad \omega \in H,$$

then  $P(H) \leq P(E_j)$ .

Note here that if  $\delta_j = 0$ , which is possible, then we are taking  $E_j = \emptyset$  and (1) and (2) will still be satisfied ( $\delta_j^{-1/p} = \infty!$ ).

Note that  $\sum \delta_j \leq 1$  so that for any  $N \in \mathbf{N}$ ,

$$\mathcal{E}' \left( \left\| \sum_{j=1}^N \delta_j^{1/p} \varepsilon_j f_{k(j),j} \right\|_q \right) \leq M\gamma^{-1}$$

and hence

$$\left\| \left( \sum_{j=1}^N \delta_j^{2/p} |f_{k(j),j}|^2 \right)^{1/2} \right\|_q \leq M\gamma^{-1}.$$

As  $(\sum \delta_j^{2/p} |f_{k(j),j}|^2)^{1/2} \geq A$  on  $E_1 \cup \dots \cup E_N$  we conclude

$$P(E_1 \cup \dots \cup E_N) \leq \left( \frac{M}{\gamma A} \right)^q \leq \frac{1}{2}\varepsilon.$$

Let  $E = \bigcup_{i=1}^\infty E_i$ . Then  $P(E) \leq \frac{1}{2}\varepsilon$ .

Now fix  $j$  so that  $n\delta_j \leq \frac{1}{2}\varepsilon$ . For each  $j$  let

$$G_j = \{\omega : \omega \notin E, |f_{ij}(\omega)| \geq A\delta_j^{-1/p}\}.$$



Then  $P(G_i) \leq \delta_i$  for  $i = 1, 2, \dots, n$ . Let  $F = \Omega \setminus (E \cup \bigcup_{i=1}^n G_i)$ . Then  $P(F) \geq 1 - \varepsilon$  and by construction

$$\|1_F \cdot f_{ij}\|_{p,\infty} \leq A, \quad i = 1, 2, \dots, n.$$

Now for any  $\alpha_1, \dots, \alpha_n$

$$\begin{aligned} \left( \mathcal{E}' \left( \left\| \sum_{i=1}^n \varepsilon_i \alpha_i w_i \right\|_q^q \right) \right)^{1/q} &\leq M(\mathcal{E}'(\|\sum \varepsilon_i \alpha_i V(e_j \otimes w_i) \cdot 1_F\|_q^q))^{1/q} \\ &= M(\mathcal{E}'(\|\sum \varepsilon_i \alpha_i f_{ij} \cdot 1_F\|_q^q))^{1/q} \\ &\leq CM \left( \sum_{i=1}^n |\alpha_i|^p \|f_{ij} \cdot 1_F\|_{p,\infty}^p \right)^{1/p} \\ &\leq ACM \left( \sum_{i=1}^n |\alpha_i|^p \right)^{1/p}. \end{aligned}$$

By a well-known result of Kahane [4], this shows that  $X$  is of type  $p$ .

LEMMA 3.2. *Suppose  $0 < q < p < 2$  and  $1 \leq M < \infty$  are given. Then there is a constant  $C = C(p, q, M)$  so that for any  $n \in \mathbb{N}$  and any linear operator  $V : l_p^{(n)} \rightarrow L_q$  which satisfies*

$$M^{-1} \|u\| \leq \|Vu\|_q \leq M \|u\|, \quad u \in l_p^{(n)},$$

then if  $f_i = Ve_i$  we have

$$C^{-1} n^{1/p} \leq \|\max |f_i|\|_q \leq C n^{1/p}.$$

PROOF. We suppose  $\{\varepsilon_i\}_{i=1}^\infty$  and  $\{\eta_i\}_{i=1}^\infty$  are two sequences of independent identically distributed random variables on some probability space. We suppose each  $\varepsilon_i$  is Bernoulli with  $P\{\varepsilon_i = +1\} = P\{\varepsilon_i = -1\} = \frac{1}{2}$ , while each  $\eta_i$  is  $r$ -stable where  $r$  is chosen so that  $p < r < 2$ . Thus

$$\mathcal{E}'(e^{i\eta}) = e^{-|\eta|^r}$$

for any  $\eta = \eta_j$ .

Now

$$\int \left| \sum_{i=1}^n \varepsilon_i f_i \right|^q dP \geq M^{-q} n^{q/p}$$

so that

$$\int \mathcal{E}'(|\sum \varepsilon_i f_i|^q) dP \geq M^{-q} n^{q/p}$$

and by Khintchine's inequality

$$\|(\sum |f_i|^2)^{1/2}\|_q \cong M^{-1} n^{1/p}.$$

Conversely

$$\begin{aligned} \mathcal{E}' \left( \int \left| \sum_{i=1}^n \eta_i f_i \right|^q dP \right) &\leq M^q (\mathcal{E}'(\sum |\eta_i|^p))^{q/p} \\ &= c_1^q M^q n^{q/p} \end{aligned}$$

where  $c_1 = \|\eta_1\|_p < \infty$ .

Hence

$$c_2^q \int (\sum |f_i|^r)^{q/r} dP \leq c_1^q M^q n^{q/p}$$

where  $c_2 = \|\eta_1\|_q < \infty$ . Thus

$$\left\| \left( \sum_{i=1}^n |f_i|^r \right)^{1/r} \right\|_q \leq c_1 c_2^{-1} M n^{1/p}.$$

Hence

$$\|\max |f_i|\|_q \leq c_1 c_2^{-1} M n^{1/p}.$$

Also

$$\sum_{i=1}^n |f_i|^2 \leq \left( \max_{i \leq n} |f_i| \right)^{2-r} \left( \sum_{i=1}^n |f_i|^r \right)$$

so that by Holder's inequality

$$\left\| \left( \sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_q \leq \left\| \left( \sum_{i=1}^n |f_i|^r \right)^{1/r} \right\|_q^{r/2} \left\| \max_{i \leq n} |f_i| \right\|_q^{1-r/2}$$

or

$$M^{-1} n^{1/p} \leq (c_1 c_2^{-1} M n^{1/p})^{r/2} \|\max |f_i|\|_q^{1-r/2}.$$

Thus

$$\|\max |f_i|\|_q \geq \left( \frac{c_2}{M c_1} \right)^{r/(2-r)} M^{-2/(2-r)} n^{1/p}$$

and the lemma is proved.

LEMMA 3.3. *Suppose  $0 < q < p < 2$  and  $0 < M < \infty$ . Then there is a constant  $\gamma = \gamma(p, q, M)$  so that whenever  $V : l_p \rightarrow L(p, \infty)$  is an embedding such that*

$$\|Vu\|_{p,\infty} \leq M\|u\|, \quad u \in l_p,$$

$$\|Vu\|_q \geq M^{-1}\|u\|, \quad u \in l_p,$$

then for any  $x \geq 1$ , if  $Ve_n = f_n$ ,

$$\liminf P(|f_n|^p > x) \geq \gamma x^{-1}.$$

PROOF. Since the inclusion  $L(p, \infty) \rightarrow L_q$  is continuous there is a constant  $C = C(p, q, M)$  so that for any finite subset  $A$  of  $N$  of cardinality  $n$

$$C^{-1}n^{1/p} \leq \left\| \max_{i \in A} |f_i| \right\|_q \leq Cn^{1/p}.$$

Choose  $\beta_1 < (1/3)^{p/q}C^{-p}$  and  $\beta_2$  so that

$$\beta_2^{1-q/p} > \frac{3q}{p-q} M^p C^q.$$

Let  $A \subset N$  be any finite subset of cardinality  $n$  and let  $g_A = \max_{i \in A} |f_i|^p$ . Then

$$\begin{aligned} P(g_A > x) &\leq \sum_{i \in A} P(|f_i|^p > x) \\ &\leq nM^p x^{-1}. \end{aligned}$$

Now

$$\begin{aligned} \int g_A^{q/p} dP &\leq (\beta_1 n)^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n) + \int_{\beta_2 n}^{\infty} \frac{q}{p} x^{(q/p-1)} P(g_A > x) dx \\ &\leq \frac{1}{3} C^{-q} n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n) + \frac{qnM^p}{p} \int_{\beta_2 n}^{\infty} x^{q/p-2} dx \\ &\leq \left( \frac{1}{3} C^{-q} + \frac{qM^p}{p-q} \beta_2^{q/p-1} \right) n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n) \\ &\leq \frac{2}{3} C^{-q} n^{q/p} + (\beta_2 n)^{q/p} P(g_A > \beta_1 n). \end{aligned}$$

However

$$\int_A g_A^{q/p} dP \geq C^{-q} n^{q/p}.$$

Hence

$$P(g_A > \beta_1 n) \geq \frac{1}{3} C^{-q} \beta_2^{-q/p}.$$

Applying this to every  $n$ -set  $A$  we see that

$$P(|f_i|^p > \beta_1 n) < \frac{1}{3n} C^{-q} \beta_2^{-q/p}$$

at most  $n - 1$  times and so

$$\liminf_{j \rightarrow \infty} P(|f_j|^p > \beta_1 n) \geq \frac{1}{3n} C^{-q} \beta_2^{-q/p}$$

for every  $n \in \mathbb{N}$  and the lemma follows for some suitable  $\gamma$ .

We can now state and prove the main theorem.

**THEOREM 3.4.** *Let  $X$  be any Banach space. Suppose  $1 \leq p < 2$ . The following conditions on  $X$  are equivalent:*

- (i)  $X$  embeds into  $L_p$ ,
- (ii)  $l_p(X)$  embeds into  $L_0$ ,
- (iii)  $L_p(X)$  embeds into  $L_0$ ,
- (iv) There is a tensor product  $l_p \otimes_\alpha X$  which is of type  $p$  and embeds into  $L_0$ .

**REMARK.** Note in the case  $p = 1$  it is not necessary to suppose the tensor product is of type  $p$  in (iv).

**PROOF.** (i)  $\Rightarrow$  (iii) If  $X$  embeds into  $L_p$  then so does  $L_p(X)$ .

(iii)  $\Rightarrow$  (ii) Immediate.

(ii)  $\Rightarrow$  (iv) This follows from Proposition 3.1 if  $p > 1$  and is obvious if  $p = 1$ .

(iv)  $\Rightarrow$  (i) We may suppose by Nikishin's theorem that we have a linear operator  $V : l_p \otimes_\alpha X \rightarrow L(p, \infty)$  so that for some  $M < \infty$  and  $q < p$  we have

$$\begin{aligned} \|Vu\|_{p,\infty} &\leq M\|u\|, \\ \|Vu\|_q &\geq M^{-1}\|u\|. \end{aligned}$$

Now define  $W : X \rightarrow \mathcal{Y}_p$  by

$$Wg = (V(e_n \otimes g))_{n=1}^\infty.$$

Then  $W$  is bounded. If  $\|g\| = 1$  and  $z \in l_p$  we have

$$\begin{aligned} \|V(z \otimes g)\|_{p,\infty} &\leq M\|z\|, \\ \|V(z \otimes g)\|_q &\geq M^{-1}\|z\|. \end{aligned}$$

Hence by Lemma 3.3 there exists  $\gamma > 0$  so that

$$\liminf_{n \rightarrow \infty} (P|V(e_n \otimes g)|^p > x) \geq \gamma x^{-1}$$

for  $x \geq 1$ .

Let  $f_n = V(e_n \otimes g)$  and  $F = (f_n) \in \mathcal{Y}_p$ . Then for  $x \geq 1$ ,

$$\begin{aligned} \lim_{n \in \mathcal{Y}} \mathcal{E}(\tau_x |f_n|^p) &= \lim_{n \in \mathcal{Y}} \int_0^x P(|f_n|^p > t) dt \\ &\geq \liminf_{n \rightarrow \infty} \int_0^x P(|f_n|^p > t) dt \\ &\geq \int_0^x \liminf_{n \rightarrow \infty} P(|f_n|^p > t) dt \\ &\geq \gamma + \gamma \log x. \end{aligned}$$

Hence

$$\phi(|F|^p) \geq \gamma$$

and in general

$$\phi(|Wg|^p)^{1/p} \geq \gamma^{1/p} \|g\|.$$

This means that  $K_p W$  maps  $X$  isomorphically into an abstract  $L_p$ -space and the result follows. See the remarks following Lemma 2.3 where  $K_p$  is defined.

EXAMPLES.  $l_p(l_q)$  ( $1 \leq p, q < 2$ ) embeds into  $L_0$  if and only if  $p \leq q$ .

#### 4. Concluding remarks

We first observe that the proofs of all the main results go through unchanged for quasi-Banach spaces. In particular we have:

**THEOREM 4.1.** *Let  $X$  be a quasi-Banach space. Then  $l_1(X)$  embeds into  $L_0$  if and only if  $X$  embeds into  $L_1$  (and hence  $X$  is locally convex).*

The analogous results for embeddability into  $L_p$  for  $0 < p < 1$  also hold.

An obvious question which arises is whether a quasi-Banach subspace of  $L_0$  of type  $p$  ( $0 < p \leq 2$ ) necessarily embeds into  $L_p$ . For  $p = 2$  this is the case, and is a simple consequence of a theorem of Kwapien [10] since every subspace of  $L_0$  has cotype 2. However for  $p = 1$  it is false; there is a non-locally convex subspace of  $L_0$  which is of type one. This is the Ribe space (see [6], [8]). The  $l_1$ -product of this space cannot be embedded in  $L_0$ . For  $0 < p < 1$  or  $1 < p < 2$  the problem is open.

A second question is whether Theorem 3.4 holds when  $p = 2$ , i.e. if  $l_2(X)$  embeds into  $L_0$  is  $X$  a Hilbert space? The methods of Proposition 3.1 only show that  $X$  is of type  $p$  for every  $p < 2$  in this case.

We shall however settle the case  $p = 2$  in the special case when  $X$  has local unconditional structure. The key is the following lemma.

LEMMA 4.1. *Let  $(u_n)$  be an unconditional basic sequence in  $L_p$  where  $0 < p < 2$ . If  $0 < q < p$  there is an unconditional basic sequence  $(v_n)$  in  $L_q$  and a constant  $C$  so that*

$$C^{-1} \left\| \sum_{n=1}^N |a_n|^{q/p} u_n \right\|_p^{p/q} \leq \left\| \sum_{n=1}^N a_n v_n \right\|_q \leq C \left\| \sum_{n=1}^N |a_n|^{q/p} u_n \right\|_p^{p/q}.$$

PROOF. Let  $\eta_n$  be any sequence of independent identically distributed  $2q/p$ -stable random variables on some probability space  $(\Omega', \Sigma', P')$  with

$$\mathcal{E}'(\exp(it\eta_n)) = \exp(-|t|^{2q/p}).$$

Consider  $g_n(\omega, \omega') = |u_n(\omega)|^{p/q} \eta_n(\omega')$  in  $L_p(\Omega \times \Omega')$ .

It is easy to see that for some constant  $C_0$

$$\left\| \sum_{n=1}^N a_n g_n \right\|_q = C_0 \left\| \left( \sum_{n=1}^N |a_n|^{2q/p} |u_n|^2 \right)^{1/2} \right\|_p^{p/q}$$

and the result follows.

REMARK. The above lemma implies that the  $q/p$ -concavification of an unconditional basic sequence in  $L_p$  embeds into  $L_q$  (we thank the referee for this remark). A similar result holds for Banach lattices embedding in  $L_p$ .

We say a Banach space  $X$  has local unconditional structure if there is a constant  $K$  so that if  $E \subset X$  and  $\dim E < \infty$  there is a finite-dimensional subspace  $F$  of  $X$  containing  $E$  with a  $K$ -unconditional basis.

THEOREM 4.2. *Let  $X$  be a Banach space with local unconditional structure. Suppose  $l_2(X)$  embeds into  $L_0$ . Then  $X$  is a Hilbert space.*

PROOF. First note that we can reduce this theorem in the case when  $X$  has an unconditional basis. Indeed for the general case one need only take any sequence  $F_n$  of finite-dimensional subspaces each with a  $K$ -unconditional basis and observe if  $Y = l_2(F_n)$  then  $Y$  has an unconditional basis and  $l_2(Y)$  embeds into  $L_0$ .

Now suppose  $X$  has an unconditional basis  $(f_n)$ , and  $l_2(X)$  embeds into  $L_p$  where  $0 < p < 2$ . Then since  $X$  has cotype 2, if  $\sum a_n f_n$  converges then  $\sum |a_n|^2 < \infty$ .

Conversely, note that there is an unconditional basic sequence  $(u_{mn})$  in  $L_p$  so that  $\sum a_{mn} u_{mn}$  converges if and only if

$$\sum_{m=1}^{\infty} \left\| \sum_{n=1}^{\infty} a_{mn} f_n \right\|^2 < \infty.$$

Fix any  $q < p$  and determine an unconditional basic sequence  $(v_{mn})$  in  $L_q$  by

Lemma 4.1. Then  $\sum a_{mn}v_{mn}$  converges if and only if

$$\sum_{m=1}^{\infty} \left\| \sum_{n=1}^{\infty} |a_{mn}|^{q/p} f_n \right\|^2 < \infty.$$

Let  $Y$  be  $[v_{1n}]$  in  $L_q$ . Then  $Y$  has an unconditional basis  $(v_{1n})$  and

$$C^{-1} \left\| \sum |a_n|^{q/p} f_n \right\|^{p/q} \leq \left\| \sum a_n v_{1n} \right\| \leq C \left\| \sum |a_n|^{q/p} f_n \right\|^{p/q}.$$

Hence  $\sum a_{mn}v_{mn}$  converges if and only if

$$\sum_{m=1}^{\infty} \left\| \sum_n a_{mn} v_{1n} \right\|_Y^{2q/p} < \infty$$

and if  $r = 2q/p$ ,  $l_r(Y)$  embeds into  $L_q$ . Hence  $Y$  embeds into  $L_r$  and is of type  $r$ . Thus if  $\sum |a_n|^2 < \infty$  then  $\sum |a_n|^{2/r} v_{n1}$  converges and hence  $\sum |a_n| f_n$  converges.

We conclude that  $(f_n)$  is equivalent to the standard basis of  $l_2$ .

#### REFERENCES

1. M. Cwikel and C. Fefferman, *Maximal seminorms on weak  $L_1$* , Studia Math. **69** (1980), 149–154.
2. M. Cwikel and Y. Sagher,  $L(p, \infty)^*$ , Indiana Univ. Math. J. **21** (1972), 781–786.
3. L. E. Dor, *On projections in  $L_1$* , Ann. Math. **102** (1975), 463–474.
4. J. P. Kahane, *Series of Random Functions*, Heath Math. Monographs, Mass., 1968.
5. N. J. Kalton, *Linear operators on  $L_p$  for  $0 < p < 1$* , Trans. Amer. Math. Soc. **259** (1980), 319–355.
6. N. J. Kalton, *Sequences of random variables in  $L_p$  for  $p < 1$* , J. Reine Angew. **329** (1981), 204–214.
7. M. Kanter, *Stable laws and embeddings of  $L_p$ -spaces*, Amer. Math. Monthly **80** (1973), 403–407.
8. J. Kupka and N. T. Peck, *The  $L_1$  structure of weak  $L_1$* , Math. Ann. **269** (1984), 235–262.
9. S. Kwapien, *Problem 3*, Studia Math. **38** (1969), 469.
10. S. Kwapien, *Isomorphic characterizations of inner product spaces by orthogonal series with vector-valued coefficients*, Studia Math. **44** (1972), 583–595.
11. H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
12. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.
13. B. Maurey, *Nouveaux théorèmes de Nikishin*, Exposés 4–5, Seminaire Maurey–Schwartz 1973–4.
14. E. M. Nikishin, *Resonance theorems and superlinear operators*, Uspekhi Mat. Nauk **25** (1970), 129–191; Russian Math. Surveys **25** (1970), 124–187.
15. E. M. Nikishin, *Resonance theorems and expansions in eigenfunctions of the Laplace operator*, Izv. Akad. Nauk SSSR Sci. Mat. **36** (1972), 795–813 = Math. USSR-Izv. **6** (1972), 788–806.
16. A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–227.
17. J. G. Pitman, *Some theorems on characteristic functions of probability distributions*, Proc. Sixth Berkeley Symposium, Vol. II, Univ. of California Press, Berkeley, 1969, pp. 393–402.
18. M. Ribe, *Examples for the nonlocally convex three space problem*, Proc. Amer. Math. Soc. **73** (1979), 351–355.